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TWO-WAREHOUSE INVENTORY MODEL FOR MAXIMUM LIFETIME ITEMS CONSIDERING CAPITAL REDUNDANCY UNDER ADVANCE PAYMENT

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ABSTRACT

Maintaining inventory of deteriorating items is an enormous task owing to their nature. These items do not only deteriorate but also expire with time. This prompt investigating both instantaneously and non- instantaneously deteriorating items that have maximum lifetime under different conditions due to their complexity. The models for those items have been developed under different promotional tools such as trade credit and discounts. However, they are yet to be explored under advance payment scheme in a capacity-constrained environment. In this work, an inventory model has been developed considering maximum lifetime items under two-warehouse. The dispatching policy adopted is First-in First-out (FIFO) due to the fact that freshness of items is considered more important than economic reasons. Advance payment was incorporated into the model to make it more practical especially in developing nations where competition to get the order from the supplier, when placed by the retailer, is high. With the help of several realistic cases, cost functions were obtained and numerical example is given as an illustration of the model. The result showed that the goods in the rented warehouse (RW) finishes at 0.746 while the replenishment cycle ends at 0.956 indicating that the retailer incurred a larger holding costs. From the sensitivity analysis, it was found that the longer the lifetime of an item, the smaller the total cost incurred by the retailer indicating that it is better for the retailer to order items with bigger lifetime from the managerial point of view.

Keywords: First-in-First-out, Maximum Lifetime Items, Two-Warehouse, Advance Payment

INTRODUCTION

The study of maximum lifetime items is crucial in the supply chain management due to the complexity of the nature of the items. To understand the nature of maximum lifetime items, several researchers have examined them under different business circumstances and various promotional policies. To date, these items have not been studied under an advance payment scheme. In this study, the items are examined under an advance payment limited to only capacity-constrained environment.

In the literature, deteriorating items have been studied under different circumstances and different promotional policies such as trade credit, different types of discounts and advance payment. In this regard, Goyal (1985) was the first to study the deteriorating items under a permissible delay in payment. Chandan & Kailash (2021) developed EOQ model for cubic deteriorating items carry forward with Weibull demand and without shortages. Molamohamadi & Mirzazadeh (2021) considered ordering policies of a deteriorating item in an EOQ model under upstream partial order-quantity-dependent trade credit and downstream trade credit. Chakraborty et al. (2018) considered two-warehouse partial backlogging inventory model with ramp type demand rate, three-parameter Weibull distribution deterioration under inflation and permissible delay in payments. Additionally, Agi & Soni (2020) developed a joint pricing and inventory decisions for perishable products with age-, stock- and price dependent demand rate. Aliyu & Sani (2022) looked at the credit-worthiness of the supplier and retailer and came up with two-warehouse inventory system model for deteriorating items considering two-level trade credit financing.

In another vein, Khan et al. (2019) studied the effect of advance payment with discount facility on supply decisions of deteriorating products whose demand is both price and stock dependent. Kumar et al. (2022) came up with two warehouse inventory model for deteriorating items with fixed shelf-life stock- dependent demand and partial backlogging. Nurhasril et al. (2023) developed a two-warehouse inventory model with rework process and time-varying demand. Pathak et al. (2024) came up with an inventory model for deteriorating goods with exponential demand, variable holding costs, and partial backlogging across two-warehouses in an inflationary environment.

Likewise, there are some inventory items that do not start to deteriorate immediately after they are kept. Rather, they maintain some freshness for a certain period before they begin to deteriorate. These types of items are termed non-instantaneous deteriorating items and have also been studied under different types of promotional policies such as Liao et al. (2022) who considered an optimal ordering policy in an EOQ model for non-instantaneous deteriorating items with defective quality and permissible delay in payments. Nath & Sen (2021) developed a completely backlogged two-warehouse inventory model for non-instantaneous deteriorating items with time and selling price dependent demand.

There are some items that do not only deteriorate continuously but also expire with time. These types of items are known as maximum lifetime items. These items were also explored under various promotional tools. Some of the studies include studies by Wu et al. (2016) that developed inventory models for deteriorating items with maximum lifetime under downstream partial trade credits to credit-risk customers by discounted cash-flow analysis. Aliyu and Sani (2021) presented two – warehouse inventory model for maximum lifetime items under partial upstream trade credit. Also, Aliyu and Adamu (2025) developed a two-warehouse inventory model for maximum lifetime items under supplier's trade credit.

In this study, maximum lifetime items have been studied considering an advance payment scheme.

The other sections are as follows: Section 2 is the methodology whereas section 3 is the model optimization. Section 4 is the result and discussion while in section 5 is the summary and conclusion.

MATERIALS AND METHODS

In this section, we formulate the model considering advance payment scheme using the mathematical function and functional relationship based on the stated assumptions.

Notations and Assumptions

Under this, the notations representing physical phenomenon that will be modelled using the assumptions are given.

The following are the notations:

$I_o(t)$, $I_r(t)$ – are the inventory levels of the own warehouse *OW* and rented warehouse *RW* respectively at time *t*.

Z – is the stocking capacity of *OW*.

I_m – is the total order quantity.

D – is the constant demand rate.

t_1, T – are the time at which inventory in *OW* and that in *RW* drop to zero respectively.

$\alpha(t)$ and $\beta(t)$ – are the deterioration rates in *OW* and *RW* respectively with $\alpha(t) = \beta(t) = {}^{1}/_{1+m-t}$, where m is the maximum lifetime period (age) of an item before it expired completely.

h_o, h_r – are the holding cost per unit item per unit time at *OW* and *RW* respectively.

A – is ordering cost per order.

c, p – are the purchasing cost per unit item and selling price of the item respectively.

TC - is the total relevant costs per year.

M - is the trade credit periods offered to the retailer by the supplier.

I_e, I_p - are the interest earned and interest payable by the retailer respectively.

The following are assumptions:

i. The model considers single items.
ii. The demand rate is assumed to be constant.
iii. The lead time is zero and shortages are not allowed.
iv. The *OW* has limited capacity and *RW* has unlimited capacity.
v. The dispatching policy is considered as FIFO because of the freshness of items.
vi. Advance payment is incorporated.

Model Formulation

Goods ordered are stocked in *OW* first with the excess going to *RW*. Because the model follows FIFO, goods are retrieved from *OW* at first. Therefore, at $t = t_1$ the inventory at *OW* drops to zero due to demand and deterioration during the period $[0, t_1]$, while goods in *RW* during that period are depleted due to deterioration only. At $t = T$, both warehouses become empty due to depletion in RW by demands and deterioration during the period$[t_1, T]$. These phenomena are represented by the following differential equations:

$$\frac{dI_o(t)}{dt} + \frac{1}{1+m-t} I_o(t) = \text{-D} \qquad 0 \le t \le t_1 \qquad (1)$$

with boundary condition $I_o(t_1) = 0$

$$\frac{dI_r(t)}{dt} + \frac{1}{1+m-t} I_r(t) = 0 \qquad 0 \le t \le t_1 \qquad (2)$$

with initial condition $I_r(0) = I_m - Z$

$$\frac{dI_r(t)}{dt} + \frac{1}{1+m-t} I_r(t) = \text{-D} \qquad t_1 < t \le T \qquad (3)$$

with boundary condition $I_r(T) = 0$

The solution of equation (1), (2) and (3) are respectively given as

$$I_o(t) = (1 + m - t)D \ln\left(\frac{1+m-t}{1+m-t_1}\right), \qquad 0 \le t \le t_1 \qquad (4)$$

$$I_r(t) = \frac{I_m - Z}{(1+m)}(1 + m - t), \qquad 0 \le t \le t_1 \qquad (5)$$

$$I_r(t) = (1 + m - t)D \ln\left(\frac{1+m-t}{1+m-T}\right) \qquad t_1 < t \le T \qquad (6)$$

To get the order quantity, we establish continuity at t_1 using equations (4) and (5). Therefore, we get

$$I_m = (1 + m)D \ln\left(\frac{1+m-t_1}{1+m-T}\right) + Z \qquad (7)$$

which is the order quantity.

Annual Total Costs

To get the annual total cost for the model, we find the following elements:

i. Annual Ordering Cost, *OC*
ii. Annual Stock Holding Cost, *HC*
iii. Annual Capital Cost Redundancy, *CR*

Ordering cost, OC

The annual ordering cost for the model is given by

$$OC = \frac{A}{T} \qquad (8)$$

Annual Stock Holding Cost, HC

The annual holding cost *HC,* is given by

$$HC = \frac{h_0}{T}\int_0^{t_1} I_0(t)dt + \frac{h_r}{T}\left[\int_0^{t_1} I_r(t)dt + \int_{t_1}^{T} I_r(t)dt\right]$$

Therefore, the total annual holding cost of the model, using equations (4), (5) and (6), is given by

$$HC = \frac{h_0}{T}\int_0^{t_1}\left[(1 + m - t)Dln\left(\frac{1+m-t}{1+m-t_1}\right)\right]dt + \frac{h_r}{T}\left[\int_0^{t_1}(1 + m - t)Dln\left(\frac{1+m-t_1}{1+m-T}\right)dt + \int_{t_1}^{T}(1 + m - t)Dln\left(\frac{1+m-t}{1+m-T}\right)dt\right] = \frac{Dh_0}{2T}\left[\frac{t_1^2}{2} - mt_1 - t_1 + (m + 1)^2 \ln\left(\frac{-m-1}{t_1-m-1}\right)\right] + \frac{Dh_r}{2T}\left[\frac{1}{2}(T^2 - t_1^2) - (T - t_1) - m(T - t_1) + (m + 1)^2 \ln\left(\frac{t_1-m-1}{T-m-1}\right)\right] \qquad (9)$$

Annual Capital Cost Redundancy, CR

Annual capital redundancy for the model is given by

$$CR = (p - c)\left[\left(Z - \frac{Z}{2}\right)T + (Z - \frac{3Z}{2})(T - t_1)\right] \qquad (10)$$

Therefore. The annual total costs for the model using equations (8), (9), and (10) is given by

$$TC = \frac{1}{T}\left\{A + \frac{Dh_0}{2}\left[\frac{t_1^2}{2} - mt_1 - t_1 + (m + 1)^2 \ln\left(\frac{-m-1}{t_1-m-1}\right)\right] + \frac{Dh_r}{2}\left[\frac{1}{2}(T^2 - t_1^2) - (T - t_1) - m(T - t_1) + (m + 1)^2 \ln\left(\frac{t_1-m-1}{T-m-1}\right)\right] + (p - c)\left[\left(Z - \frac{Z}{2}\right)T + (Z - \frac{3Z}{2})(T - t_1)\right]\right\} \qquad (11)$$

Inventory Optimization

Since we are dealing with decision variables t_1 and T, gradient method is applied to establish the necessary and sufficient conditions for the existence and uniqueness of the optimal solutions.

The necessary conditions for TC to be minimized are given by $\frac{\partial TC}{\partial t_1} = 0$ and $\frac{\partial TC}{\partial T} = 0$

Using (11), we find that

$$\frac{\partial TC}{\partial t_1} = \frac{1}{T}\left\{\frac{Dho+Dhr}{2(t_1-m-1)}\left(t_1(t_1 - 2m - 2)\right) - (p - c)\left(Z - \frac{3Z}{2}\right)\right\}$$

And setting the derivative to zero, we see that

$$\frac{1}{T}\left\{\frac{Dho+Dhr}{2(t_1-m-1)}\left(t_1(t_1 - 2m - 2)\right) - (p - c)\left(Z - \frac{3Z}{2}\right)\right\} = 0 \qquad (12)$$

Also,

$$\frac{\partial TC}{\partial T} = \frac{1}{T}\left\{\frac{Dhr}{2}\left(\frac{T(T-2m-2)}{T-m-1}\right) + (p-c)\left(\left(Z-\frac{Z}{2}\right) + \left(Z-\frac{3Z}{2}\right)\right) - TC\right\}$$

And setting the derivative to zero, we see that

$$\frac{1}{T}\left\{\frac{Dhr}{2}\left(\frac{T(T-2m-2)}{T-m-1}\right) + (p-c)\left(\left(Z-\frac{Z}{2}\right) + \left(Z-\frac{2Z}{2}\right)\right) - TC\right\} = 0 \quad (13)$$

The solution to the highly non-linear equations (12) and (13) give the values of t_1 and T. To confirm that the optimal solution (t_1^*, T^*), exist and unique, we show that,

$$\left[\left(\frac{\partial^2 TC}{\partial T^2}\right)\left(\frac{\partial^2 TC}{\partial t_1^2}\right) - \left(\frac{\partial^2 TC}{\partial T \partial t_1}\right)\left(\frac{\partial^2 TC}{\partial t_1 \partial T}\right)\right]\Big|_{(t_1^*, T^*)} > 0 \quad (14)$$

Using equation (12) when evaluated at (t_1^*, T^*)

$$\frac{\partial^2 TC}{\partial t_1^2} = \frac{1}{T}\left\{Dhr + Dho\left(\frac{t_1^2 - 2mt_1 - 2m^2 + 2m + 2}{(t_1 - m - 1)^2}\right)\right\} > 0 \quad (15)$$

And using (13)

$$\frac{\partial^2 TC}{\partial T^2} = \frac{1}{T}\left\{\frac{Dhr}{2}\left(\frac{(T-m-1)^2 + (m+1)^2}{(T-m-1)^2}\right) - 2\frac{\partial TC}{\partial T}\right\} > 0 \quad (16)$$

since $\frac{\partial TC}{\partial T} = 0$

Using equation (12), we obtained

$$\frac{\partial^2 TC}{\partial T \partial t_1} = -\frac{1}{T}\left\{\frac{\partial TC}{\partial t_1}\right\} = 0 \quad (17)$$

Using equation (13) $\frac{\partial^2 TC}{\partial t_1 \partial T}$ gives,

$$\frac{\partial^2 TC}{\partial t_1 \partial T} = \frac{1}{T}\left\{-\frac{\partial TC}{\partial t_1}\right\} = 0 \quad (18)$$

Theorem 1**:** If the TC given in equation (11) satisfies equation (14), then *TC* is convex.

Proof:

From equations (15), (16), (17) and (18), we see that equation (14) is greater than zero, which implies that the Hessian matrix is positive definite. Hence, TC is convex. Proved

RESULTS AND DISCUSSION

Numerical example is presented below to illustrate the model.

Example 1: Given the parameters, $A = ₦1000$, $D = 1500, m = 1yr, Z = 10000, p = ₦15$ per unit item, $c = ₦10$ per unit item, $h_r = ₦3$ per unit, $h_o = ₦1$ per unit

Using the parameters above, MATLAB software was used to obtain the following result:

Table 1: Result of the Example

t_1	T	TC
0.746	0.956	₦21240.673

This shows that if the goods in the *OW* are finished at 0.746 (272 days) and the overall quantity of goods is finished at 0.956 (349 days), then the retailer will incur ₦21240.673

Sensitivity Analysis

Using the above example, we study the effect of some parameters on the optimal values of the t_1, T and TC. the results of the sensitivity analysis are shown below:

Table 2: Result of the Sensitivity Analysis

Parameter	% change in parameter	New t_1	% in t_1	New T	% in T	New TC	% in TC
	+50%	1.078	7.8	15.538	1525.314	22499.431	5.926
	+25%	1.077	44.370	15.553	0.153	22499.604	5.927
A	0	0.746	0	0.956	0	21240.673	0
	-25%	0.743	-0.402	0.938	-1.883	21228.783	-0.056
	-50%	0.740	-10.858	0.920	-3.766	21216.518	-0.114
	+50%	0.665	-10.858	0.794	-16.946	33718.308	58.744
	+25%	1.531	105.228	24.315	2443.410	28828.997	35.725
Z	0	0.746	0	0.956	0	21240.673	0
	-25%	1.075	44.102	1.463	53.033	15200.672	-28.436
	-50%	3.735	400.670	0.245	-74.372	10030.983	-52.775
	+50%	5.582	648.257	6.735	604.498	22398.845	5.423
	+25%	0.347	-53.485	0.474	-50.418	20891.769	-1.643
m	0	0.746	0	0.956	0	21240.673	0
	-25%	0.543	-27.212	0.728	-23.849	21305.482	0.305
	-50%	0.330	-55.764	0.467	-51.151	21414.367	0.815

From the result of the sensitivity analysis:

For the parameter *A*, increase in the value of *A* leads to increase in t_1, significant increase *T* that sharply decrease as *A* increase further, and slight increase in TC that later decrease slightly. This does not agree with common sense. However, decrease in the values of *A* decreases the values of t_1, T and TC which agrees with common sense.

For the parameter *Z*, increase in the value of *Z* first leads to an increase in t_1 and later decrease, significant increase *T* that sharply decrease as *Z* increases further and as well leads to significant increase in *TC*. This agrees with common sense. However, decrease in the values of *Z* increases the values of t_1, at first increase in value of *T* and later decrease with further increase with *Z* and decreases in values of *TC* which agree with common sense.

Also, for the parameter *m*, increase in the value of *m* led to instability in values of t_1, *T* and *TC*. This does not agree with common sense. However, decrease in the values of m decreases the values of t_1, and *T* while it showed increases in values of TC.

CONCLUSION

This study presents a challenging scenario in the literature of inventory control theory by studying maximum lifetime items considering an advance payment scheme. The total cost was obtained and shown to be convex. The result indicates that the retailer should be dealing with a higher quantity of goods in order to reduce the total cost although, this will result in longer replenishment circle.

It is recommended that this study be extended to consider more realistic demand patterns. Shortages can be incorporated to make it more applicable in the real market. For the dispatching policy, random dispatching policy can also be considered instead of the *FIFO* or *LIFO* dispatching approaches.

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